Various Functions Solutions

1. We can use plug-in method to solve this question.

   Since $g(a,b) = f(a) + f(b)$, then:
   $g(a+b,a+b) = f(a+b) + f(a+b) = 2f(a+b)$;
   $g(a,a) + g(b,b) = f(a) + f(a) + f(b) + f(b) = 2f(a) + 2f(b)$.

   Thus the question asks: for which function f below will $2f(a+b) = 2f(a) + 2f(b)$ --> $f(a+b) = f(a) + f(b)$?

   Say a=-1 and b=1, then the question becomes: for which function f below will $f(0) = f(-1) + f(1)$.

   A: x +3
   $f(0) = 0 + 3 = 3$
   $f(-1) + f(1) = (-1+3) + (1+3) = 6$
   No match.

   B: x^2
   $f(0) = 0^2 = 0$
   $f(-1) + f(1) = (-1)^2 + (1)^2 = 2$
   No match.

   C: |x|
   $f(0) = |0| = 0$
   $f(-1) + f(1) = |-1| + |1| = 2$
   No match.

   D: 1/x
   $f(0) = \frac{1}{0} = undefined$
   $f(-1) + f(1) = \frac{1}{-1} + \frac{1}{1} = 0$
   No match.

   E: x/4
   $f(0) = \frac{0}{4} = 0$
   $f(-1) + f(1) = \frac{-1}{4} + \frac{1}{4} = 0$
   Match.

   Answer: E.

2. Notice that the greatest common factor of 10 and x, GCF(10,x), naturally must be a factor of 10: 1, 2, 5, and 10. Thus from f(10,x)=11 we can get four different values of x:

GCF(10,x)=1 --> $f(10,x)=11=\frac{10+x}{1}$ --> $x=1$;
GCF(10,x)=2 --> $f(10,x)=11=\frac{10+x}{2}$ --> $x=12$;
GCF(10,x)=5 --> $f(10,x)=11=\frac{10+x}{5}$ --> $x=45$;
GCF(10,x)=10 --> $f(10,x)=11=\frac{10+x}{10}$ --> $x=100$.

(1) x is a square of an integer --> x can be 1 or 100. Not sufficient.

(2) The sum of the distinct prime factors of x is a prime number ---> distinct primes of 12 are 2 and 3: $2+3=5=prime$, distinct primes of 45 are 3 and 5: $3+5=8\neq prime$ and distinct primes of 100 are 2 and 5: $2+5=7=prime$. x can be 12 or 100. Not sufficient.

(1)+(2) x can only be 100. Sufficient.

Answer: C.

3. **Answer is C**

While we spend a lot of time honing the skill of translating English into algebra, there is sometimes great comfort to be gained in a sequence problem through doing precisely the reverse. The general rule for this sequence is that we derive each term based on the two terms that precede it – specifically by subtracting the absolute value of the previous term from the absolute value of the "pre-previous" one.

So $a_3=|a_1|-|a_2|$, and $a_4=|a_2|-|a_3|$, and $a_5=|a_3|-|a_4|$, and so on. We are told that the sequence begins 0, 3, ..., so we can derive that:

$a_3=|a_1|-|a_2|=|0|-|3|=0-3=-3$.
Then $a_4=|a_2|-|a_3|=|3|-|-3|=3-3=0$.
Then $a_5=|a_3|-|a_4|=|-3|-|0|=3-0=3$.

As soon as we've seen a_4 and a_5 turn out to be 0 and 3 consecutively, we know that the next number to show up in the sequence will be identical to the number that showed up after the last time we saw 0 and 3 appear consecutively (i.e. as a_1 and a_2) — so a_6 will be identical to a_3, so -3. So we have established the pattern 0, 3, -3, 0, 3, -3, ... for our sequence.

Every time we finish a complete cycle within the sequence, the sum returns to 0 (since 0 + 3 + -3 = 0). We finish a cycle after every third entry (i.e. after the third, the sixth, the ninth, and so on), so we will have done so (and returned our running sum to 0) with a_{99}. a_{100}will then add itself (0) to that sum, and a_{101} will add itself (3) onto that. So we will land at a sum of 3 for s_{101}.

4.

In all the option just put 2-x in place of x

A. f(x)=x+2===>f(2-x)=2-x+2=4-x==>not equal

B. (fx)=2x-x^2====>f(2-x)=2(2-x)-(2-x)^2=4-2x-x^2+4x-4=2x-x^2===>equal==correct

C. f(x)=2-x===>f(2-x)=2-(2-x)=x===>not equal

D. f(x)=(2-x)^2===>f(2-x)=(2-2+x)^2=x^2===>not equal

E. f(x)=x^2===>f(2-x)=(2-x)^2===>not equal

5. $h(100)+2=2*4*6*...*100+2$. Notice that we can factor out 2 from h(100)+2, thus the smallest prime factor of h(100)+2 is 2: $h(100)+2=2*(4*6*...*100+1)$.

Answer: E.

6. A. $f(a+b)=(a+b)^2=a^2+2ab+b^2\neq f(a)+f(b)=a^2+b^2$

B. $f(a+b)=(a+b)+1\neq f(a)+f(b)=a+1+b+1$

C. $f(a+b)=\sqrt{a+b}\neq f(a)+f(b)=\sqrt{a}+\sqrt{b}$.

D. $f(a+b)=\frac{2}{a+b}\neq f(a)+f(b)=\frac{2}{a}+\frac{2}{b}$.

E. $f(a+b)=-3(a+b)=-3a-3b=f(a)+f(b)=-3a-3b$. Correct.

Answer: E.

OR, as f(a+b)= f(a)+f(b) must be true for all positive numbers a and b, then you can randomly pick particular values of a and b and check for them:

For example: $a=2$ and $b=3$

A. $f(a+b)=f(5)=5^2=25\neq f(a)+f(b)=f(2)+f(3)=2^2+3^2=13$

B. $f(a+b)=f(5)=5+1=6\neq f(a)+f(b)=f(2)+f(3)=(2+1)+(3+1)=7$

C. $f(a+b)=f(5)=\sqrt{5}\neq f(a)+f(b)=f(2)+f(3)=\sqrt{2}+\sqrt{3}$

D. $f(a+b)=f(5)=\frac{2}{5}\neq f(a)+f(b)=f(2)+f(3)=\frac{2}{2}+\frac{2}{3}=\frac{5}{3}$

E.
$f(a+b)=f(5)=-3*(5)=-15=f(a)+f(b)=f(2)+f(3)=-3*(2)-3*(3)=-15$
. Correct.

It might happen that for some choices of a and b other options may be "correct" as well. If this happens just pick some other numbers and check again **these "correct" options only**.

7. $f(x)=\text{"}some\ expression\ with\ variable\ x\text{"}$, means that the value of $f(x)$ can be found by calculating the expression for the particular x.

For example: if $f(x)=3x+2$, what is the value of $f(3)$? Just plug 3 for x, $f(3)=3*3+2=11$, so if the function is $f(x)=3x+2$, then $f(3)=11$.

There are some functions for which $f(x)=f(-x)$. For example: if we define $f(x)$ as $f(x)=3*x^2+2$, the value of $f(x)$ will be always positive and will give the following values: for $x=-5$, $f(x)=3*(-5)^2+2=77$;
for $x=0$, $f(0)=3*0^2+2=2$. Please note that $f(x)$ in this case is equal to $f(-x)$, meaning that for positive values of x you'll get the same values of $f(x)$ as for the negative values of x.

So, basically in original question we are told to define the expression, for which $f(x)=f(1-x)$, which means that plugging x and $1-x$ in the expression must give same result.

A. $f(x)=1-x$ --> $1-x$ is the expression for $f(x)$, we want to find whether the expression for $f(1-x)$ would be the same: plug $1-x$ --> $f(1-x)=1-(1-x)=x$.
As $1-x$ and x are different, so $f(x)$ does not equal to $f(1-x)$.

The same with the other options:

(A) $f(x)=1-x$, so $f(1-x)=1-(1-x)=x$ --> $1-x$ and x: no match.

(B) $f(x)=1-x^2$, so $f(1-x)=1-(1-x)^2=1-1+2x-x^2=2x-x^2$ --> $1-x^2$ and $2x-x^2$: no match.

(C) $f(x)=x^2-(1-x)^2=x^2-1+2x+x^2=2x-1$, so $f(1-x)=2(1-x)-1=1-2x$ --> $2x-1$ and $1-2x$: no match.

(D) $f(x)=x^2*(1-x)^2$, so $f(1-x)=(1-x)^2*(1-1+x)^2=(1-x)^2*x^2$ --> $x^2*(1-x)^2$ and $(1-x)^2*x^2$. Bingo! if $f(x)=x^2*(1-x)^2$ then $f(1-x)$ also equals to $x^2*(1-x)^2$.

Still let's check (E)

(E) $f(x)=\frac{x}{1-x}$ --> $f(1-x)=\frac{1-x}{1-1+x}=\frac{1-x}{x}$. $\frac{x}{1-x}$ and $\frac{1-x}{x}$: no match.

But this problem can be solved by simple number picking: plug in numbers.

As stem says that "following functions f is f(x) = f (1-x) **for all x**", so it should work for all choices of x.

Now let x be 2 (note that: -1, 0, and 1 generally are not good choices for number picking), then $1-x=1-2=-1$. So we should check whether $f(2)=f(-1)$.

(A) $f(2)=1-x=1-2=-1$ and $f(-1)=1-(-1)=2$ --> $-1\neq2$;

(B) $f(2)=1-x^2=1-4=-3$ and [illegible] --> $-3\neq0$;

(C) $f(2)=x^2-(1-x)^2=x^2-1+2x+x^2=2x-1=2*2-1=3$ and $f(-1)=2*(-1)-1=-3$ --> $3\neq-3$;

(D) $f(2)=x^2*(1-x)^2=(-2)^2*(-1)^2=4$ and $f(-1)=(-1)^2*2^2=4$ --> $4=4$, correct;

(E) $f(2)=\frac{x}{1-x}=\frac{2}{1-2}=-2$ and $f(-1)=\frac{-1}{1-(-1)}=-\frac{1}{2}$ --> $-2\neq-\frac{1}{2}$.

It might happen that for some choices of x other options may be "correct" as well. If this happens, just pick some other number for x and **check again these "correct" options only**.

8. Let the 3-digit positive integer m be abc and positive integer v be rst.
Question: $(100a+10b+c)-(100r+10s+t)=?$.

Then $f(m)=2^a*3^b*5^c$ and $f(v)=2^r*3^s*5^t$.

Given: $f(m)=9*f(v)$ --> $2^a*3^b*5^c=9*2^r*3^s*5^t$ --

> $2^{a-r}*3^{b-s}*5^{c-t}=3^2$ (or $2^a*3^b*5^c=2^r*3^{s+2}*5^t$).

So, $a-r=0$, $b-s=2$ and $c-t=0$ (or $a=r$, $b=s+2$ and $c=t$) --> $m-v=(100a+10b+c)-(100r+10s+t)=10b-10(b-2)=20$. For example if m and v are 143 and 123 respectively (hundreds and units digit are equal and tens digit of m is 2 more than tens digit of v) then $143-123=20$.

Answer: D.

9. In order the answer to be A, the question should read:

If $f(x)=\frac{125}{x^3}$, what is the value of $f(5x)*f(x/5)$ in terms of f(x)?

Say $x=1$, then:

$f(x)=f(1)=\frac{125}{1^3}=125$;

$f(5x)=f(5)=\frac{125}{5^3}=1$;

$f(\frac{x}{5})=f(\frac{1}{5})=\frac{125}{(\frac{1}{5})^3}=125*125$.

$f(5x)*f(\frac{x}{5})=f(5)*f(\frac{1}{5})=125^2$ and since $f(x)=f(1)=125$, then $f(5x)*f(\frac{x}{5})=125^2=(f(x))^2$.

Answer: A.

10. Given: $f(x)=-\frac{1}{x}$.

As $f(a)=-\frac{1}{a}=-\frac{1}{2}$ then $a=2$, also as $f(ab)=-\frac{1}{ab}=\frac{1}{6}$ then $ab=-6$ --> $2b=-6$ --> $b=-3$.

Answer: D.

11. Given: $f(n)=(n+4)(n+5)(n+6)$, where n is a positive integer.
Question: $f(n)$ **must** be divisible by which one of the following numbers.

Now, $(n+4)(n+5)(n+6)$ is the product of 3 consecutive integers so out of them one is definitely divisible by 3 and at least one is divisible by 2, so $f(n)$ must be divisible by 2*3=6.

Answer: C.

Generally out of ANY k consecutive integers one is always divisible by k and at least one by $k-1$, $k-2$, ... For example out of ANY 5 consecutive integers there is one which is divisible by 5, and at least one which is divisible by 4, 3, and 2. That's because an integer divided by an integer k can give a remainder of: 0 (when it's divisible by k), 1, 2, ..., or $k-1$ (total of k different remainders from 0 to $k-1$), so out of k consecutive integers there definitely will be one which gives a reminder of zero, so divisible by k.

Which give us the following property: the product of k consecutive integers is always divisible by $k!$, so by k too. For example: given $k=4$ consecutive integers $\{3,4,5,6\}$ --> the product of 3*4*5*6 is 360, which is divisible by 4!=24.

If we apply this property to the original question we'll have that the product of given 3 consecutive integers $(n+4)(n+5)(n+6)$ must be divisible by 3!=6.

12. The function basically transforms the digits of integer n into the power of primes: 2, 3, 5, ...

For example:
$p(9)=2^9$;
$p(49)=2^9*3^4$;
$p(349)=2^9*3^4*5^3$;
$p(6349)=2^9*3^4*5^3*7^4$;
...

The question asks for the leas number that cannot be expressed by the function p(n).

So, the digits of n transform to the power and since single digit cannot be more than 10 then p(n) cannot have the power of 10 or higher.

So, the least number that cannot be expressed by the function p(n) is $2^{10}=1,024$ (n just cannot have 10 as its digit).

Answer: D.

13. This question for instance basically asks: how many positive integers are less than given prime number p which have no common factor with p except 1.

Well as p is a prime, all positive numbers less than p have no common factors with p (except common factor 1). So there would be p-1 such numbers (as we are looking number of integers less than p).

For example: if p=7 how many numbers are less than 7 having no common factors with 7: 1, 2, 3, 4, 5, 6 --> 7-1=6. Answer : A.

14. The function F(n) is defined as the product of all the consecutive positive integers between 1 and n^2, inclusive, thus $F(3)=1*2*3*...*9=9!$.

The function G(n) is defined as the product of the *squares* of all the consecutive positive integers between 1 and n, inclusive, thus $G(3)=1^2*2^2*3^2=3!*3!$.

$\frac{F(3)}{G(3)}=\frac{9!}{3!*3!}=\frac{4*5*6*7*8*9}{6}=2^2*5*7*2^3*9=2^5*(5*7*9)$.

The power of 2 is 5.

Answer: E.

15. f(24) = 2*4*6*8*10*12*14*16*18*20*22*24 = 2^12*(1*2*3*4*5*6*7*8*9*10*11) --> the greatest prime factor is 11.

Answer: E.

16.

Given that:
$V(a,a)=a$;
$V(a,b)=V(b,a)$;
$V(a,a+b)=(1+\frac{a}{b})V(a,b)$.

Question asks to find the value of $V(66,14)$.

Notice that only the first function gives answer as a simple value rather than another function, thus we should manipulate with $V(66,14)$ so that to get $V(a,a)=a$ in the end.

$V(66,14)=V(14,66)=V(14,14+52)$;

$V(14,14+52)=(1+\frac{14}{52})V(14,52)=\frac{33}{26}*V(14,14+38)$;

$\frac{33}{26}*V(14,14+38)=\frac{33}{26}*(1+\frac{14}{38})V(14,38)=\frac{33}{19}*V(14,14+24)$;

$\frac{33}{19}*V(14,14+24)=\frac{33}{19}*(1+\frac{14}{24})V(14,24)=\frac{33}{12}*V(14,14+10)$;

$\frac{33}{12}V(14,14+10)=\frac{33}{12}*(1+\frac{14}{10})V(14,10)=\frac{33}{5}*V(10,14)=\frac{33}{5}*V(10,10+4)$;

$$\frac{33}{5}V(10,10+4)=\frac{33}{5}*(1+\frac{10}{4})V(10,4)=\frac{33*7}{5*2}*V(4,10)=\frac{33*7}{5*2}*V(4,4+6)$$;

$$\frac{33*7}{5*2}*V(4,4+6)=\frac{33*7}{5*2}*(1+\frac{4}{6})V(4,6)=\frac{33*7}{2*3}*V(4,4+2)$$;

$$\frac{33*7}{2*3}*V(4,4+2)=\frac{33*7}{2*3}*(1+\frac{4}{2})V(4,2)=\frac{33*7}{2}*V(2,4)=\frac{33*7}{2}*V(2,2+2)$$;

$$\frac{33*7}{2}*V(2,2+2)=\frac{33*7}{2}*(1+\frac{2}{2})V(2,2)=\frac{33*7}{2}*2*2=462$$.

Answer: E.

17. (1) f(2) = 100 --> $f(2)=100=a^2$ --> $a=10$ or $a=-10$. Two answers, not sufficient.

(2) f(3) = -1,000 --> --> $f(3)=-1{,}000=a^3$ --> only one solution: $a=-10$. Sufficient.

Answer: B.

18. (1) f(2) = 100 --> $f(2)=100=a^2$ --> $a=10$ or $a=-10$. Two answers, not sufficient.

(2) f(3) = -1,000 --> --> $f(3)=-1{,}000=a^3$ --> only one solution: $a=-10$. Sufficient.

Answer: B.